Math-601D-201: Lecture 22. Geometry of pseudo-convex domains

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Reminder on pseudo-convex domains

$\Omega \subset \mathbb{C}^n$ connected open set.

Theorem

$\Omega \subset \mathbb{C}^n$. The following are equivalent.

- *Ω is pseudo-convex;*
- *$H^{p,q}(\Omega) = 0$ for all $q > 0$;*

Extension of holomorphic functions

$\Omega \subset \mathbb{C}^n$

Theorem

Suppose that $H^{0,1}(\Omega) = 0$*, and pick* $h \in \mathcal{O}(\Omega)$ *such that* $dh|_{\{h=0\}} \neq 0$*. Write* $M = \{h = 0\}$*. Then the restriction morphism*

$$\mathcal{O}(\Omega) \to \mathcal{O}(M)$$

is surjective.

Extension of holomorphic functions (complement)

$\longrightarrow$ using L^2-technics, one can prove a far-reaching generalization of the previous result, see `https://arxiv.org/1407.4946`.

Theorem (Ohsawa-Takegoshi's theorem)

$u \in PSH(\Omega)$. $\exists C = C(\Omega)$ such that for any $f \in \mathcal{O}(M) \cap L^2(e^{-u})$, there exists $F \in \mathcal{O}(\Omega) \cap L^2(e^{-u})$ such that $F|_M = f$ and

$$\int |F|^2 e^{-u} \leq C \int |f|^2 e^{-u}$$

Topology of pseudo-convex domains

$\Omega \subset \mathbb{C}^n$ pseudo-convex.

Theorem

$$H^q(\Omega, \mathbb{C}) \equiv \{\partial - \textit{ closed hol. } q \textit{ forms}\} / \partial \{\textit{hol. } q-1 \textit{ forms}\}$$

In particular

$$H^q(\Omega, \mathbb{C}) = 0 \textit{ for all } q > n$$

$\longrightarrow H^n(\Omega, \mathbb{C}) = 0$ for any domain

$\longrightarrow \Omega$ pseudo-convex has the homotopy type of a CW-complex of dimension n

Final remarks: Stein manifolds

Definition (Stein manifold)

A complex manifold M of dimension n is Stein iff

1. *holomorphically convex*

$$\hat{K} = \{ z \in M, |f(z)| \leq \sup_K |f|, \forall f \in \mathcal{O}(M)\}$$

   is compact for all compact K.

2. *holomorphic functions separate points: for all $z_1 \neq z_2$, there exists $f \in \mathcal{O}(M)$ s.t. $f(z_1) \neq f(z_2)$*
3. *for any $z \in M$, there exists $f_1, \cdots, f_n \in \mathcal{O}(M)$ such that $f = (f_1, \cdots, f_n)$ is a local diffeomorphism at z.*

Stein manifolds

Theorem

M is Stein iff $H^q(M, \mathcal{F}) = 0$ for all coherent sheaf $\mathcal{F}$ and for all $q > 0$.

Theorem

M is Stein iff there exists a proper analytic embedding $M \hookrightarrow \mathbb{C}^N$

$\rightsquigarrow$ Stein manifolds play the role of affine varieties in analytic geometry.